OPTIMIZATION OF A CONVECTION-COOLED COOLING FIN

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We consider the solution of the variational problem of defining the form of a narrow fin of least cross-sectional area for the convection dissipation of a given heat flux. The effect of certain parameters on the optimum shape of the fin is analyzed.

The optimization of fins for convective heat transfer was the subject of several papers [1–3, 6]. It was shown that fins of variable thickness ensure a greater rate of heat removal than rectangular fins, and the profiles of such fins were determined on certain arbitrary assumptions. In this paper we consider, as a follow-up to [4], the weight optimization of straight fins cooled by convection.

Let us consider the cross section of a narrow fin heated at its root OA (Fig. 1a). The shape OAB of this fin yielding the minimum cross-sectional area is to be determined for a given input heat flux Q_0 and temperature T_0. The effect of various factors on the length and the area of the cross section of the fin is also to be established. The input equations are those of thermal conductivity and of heat transfer per unit of fin length:

$$Q = 2\lambda y \frac{dT}{dx}, \qquad (1)$$

$$\frac{dQ}{dx} = 2\alpha (T - T_{\text{m}}) \sqrt{1 + y'^2}. \qquad (2)$$

The boundary conditions are:

$$x = 0, \quad Q = Q_0, \quad T = T_0;$$

$$x = L, \quad Q = 0, \quad T = T_L, \quad y = y_{\min}.$$

The area of the fin cross section is

$$F = -2 \int_0^L y dx. \qquad (3)$$

Let us use dimensionless variables defined by the relationships:

$$q = \frac{Q}{Q_0}; \quad \theta = \frac{T - T_{\text{m}}}{T_0 - T_{\text{m}}}; \quad z = \frac{x}{Q_0/2\alpha (T - T_{\text{m}})};$$

$$u = \frac{y}{Q_0^2/4\lambda\alpha (T_0 - T_{\text{m}})^2}; \quad f = \frac{F}{Q_0^3/8\alpha^2\lambda (T_0 - T_{\text{m}})^3}.$$

Since a narrow fin is considered, y' may be neglected as small in comparison to the unit in formula (2). This necessitates the fulfillment of the condition

$$y'^2 \ll 1. \qquad (4)$$

Equations (1)–(3) written in dimensionless form are:

$$q = u \frac{d\theta}{dz}, \qquad (5)$$

$$\frac{dq}{dz} = \theta, \qquad (6)$$

$$f = -2 \int_0^l u dz, \qquad (7)$$

where for $z = 0$, $q = 1$, and $\theta = 1$; for $z = l$, $q = 0$, $\theta = \theta_l$, and $u = u_{\min}$.

Mathematically the problem is formulated thus: we have to find that function $u = u(z)$ which would reduce functional (7) to its minimum, and then solve Eqs. (5) and (6) for the derived functional relationship $u = u(z)$.

To solve this problem we use Pontryagin's maximum principle [5]. We express the temperature θ, the coordinate z, and the area f similarly to [4] in terms of the variable q:

$$\frac{d\theta}{dq} = \frac{q}{u\theta}, \qquad (8)$$

$$du = \frac{dq}{\theta}, \qquad (9)$$

$$f = -2 \int_1^0 \frac{u dq}{\theta}. \qquad (10)$$

We construct for this system the Hamilton function (see [5])

$$H = \psi_1 \frac{q}{u\theta} - \psi_2 2 \frac{u}{\theta}, \qquad (11)$$

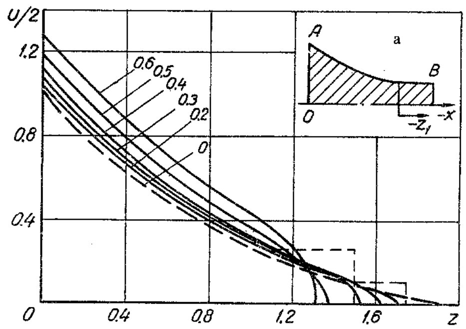

Fig. 1. Profiles of narrow fins for various temperatures of the fin tip, and the outline (a) of an optimum fin. (Numerals indicate values of θ_l for each curve. Rectangular parts of fins for $u_{\min} = 0.5$ and $u_{\min} = 0.2$ are indicated by dashed lines.)

where

$$\frac{d\psi_1}{dq} = -\frac{\partial H}{\partial \theta} = \psi_1 \frac{q}{u\theta^2} - \psi_2 \frac{2u}{\theta^2}; \qquad (12)$$

$$\frac{d\psi_2}{dq} = -\frac{\partial H}{\partial f_i} = 0. \qquad (13)$$

For functional (10) to reach its minimum it is necessary according to [5] that in the domain of acceptable values of u the Hamiltonian H attain its maximum with respect to u for any arbitrary θ and ψ_1. This domain consists of all $u > u_{min}$.

It will be seen from (13) that function ψ_2 = const. From the condition of existence of a minimum of functional (10) it follows that $\psi_2 \geq 0$, and from the transversality and the boundary conditions we have, respectively, $\psi_2 = 1$ for $q = 0$, and $u = u_{min}$. It follows from (11) that H can attain its maximum for $u = u_{min}$, provided

$$\sqrt{-\frac{q\psi_1}{2}} < u_{\min}. \qquad (14)$$

Hence the optimum shape is a straight line, and part of the fin is of a rectangular form. At a certain $q = q^*$ the inequality sign in (14) is reversed. Function H attains its maximum at

$$\psi_1 = -\frac{2u^2}{q}. \qquad (15)$$

This equation together with Eqs. (12) and (8) yields the problem solution:

$$\psi_1 = C_1\theta^2, \qquad (16)$$

where C_1 is the arbitrary constant determined from the boundary conditions.

Substituting (16) into (15), we obtain for the dependence of u on the dimensionless temperature θ that

$$u = \sqrt{-\frac{C_1 q}{2}}\,\theta. \qquad (17)$$

Solving together Eqs. (17) and (8) and using the boundary conditions, we derive the relationship between the heat flux q and the temperature θ:

$$\theta = [\theta_l^3 + (1-\theta_l^3)\, q^{3/2}]^{1/3}, \qquad (18)$$

where θ_l is the temperature of the fin tip with $u_{min} = 0$. The coordinates of the fin cross section are defined by:

$$z = \int_1^q \frac{dq}{\theta} = \int_1^q \frac{dq}{[\theta_l^3 + (1-\theta_l^3)\, q^{3/2}]^{1/3}}, \qquad (19)$$

$$u = \frac{2q^{1/2}}{1-\theta_l^3}\, [\theta_l^3 + (1-\theta_l^3)\, q^{3/2}]^{1/3}. \qquad (20)$$

Contours of the fin cross section calculated by Eqs. (19) and (20) are shown in Fig. 1.

The analytical relationship between z and u can be established by setting $\theta_l = 0$

$$u = 2\left(1 + \frac{z}{2}\right)^2, \quad z < 0.$$

The dimensionless length of the fin is $l = 2$, and the temperature variation along it is linear: $\theta = 1 + z/2$.

A fin having temperature $\theta_l = 0$ has the least cross-sectional area of all optimum fins. In fact, if we calculate the cross-sectional area by formula (10), we obtain

$$f = -2\int_1^0 \frac{2q^{1/2}\,\theta\, dq}{(1-\theta_l^3)\,\theta} = \frac{8}{3}\,\frac{1}{1-\theta_l^3}. \qquad (21)$$

Thus a fin whose tip is maintained at temperature $\theta_l = 0$ has the least cross-sectional area. The dependence of the area of the fin cross section on the temperature of its tip is shown in Fig. 2. The thickness of the fin root and the steepness u' of its slope are also affected by the temperature θ_l. Since $q = 1$ at the root of the fin, the effect of θ_l on u is defined, in accordance with Eq. (20), by

$$u = \frac{2}{1-\theta_l^3}.$$

Let us determine the slope u' of the fin contour

$$\frac{du}{dz} = \frac{du}{dq}\,\frac{dq}{dz} = \frac{1}{1-\theta_l^3}\,\frac{\theta_l^3 q^{-1/2} + 2q(1-\theta_l^3)}{[\theta_l^3 + (1-\theta_l^3)\, q^{3/2}]^{1/3}}. \qquad (22)$$

At the root the slope of the fin contour and its thickness increase with increasing θ_l:

$$\frac{du}{dz} = 1 + \frac{1}{1-\theta_l^3}.$$

Let us verify the limits of applicability of assumption (4). For simplicity we assume $\theta_l = 0$ for which $du/dz = 2$. Transforming to dimensional quantities

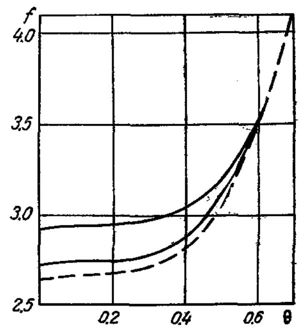

Fig. 2. Dependence of the fin cross-sectional area on temperature θ_l. The upper solid curve is for $u_{min} = 0.5$, and the lower one for $u_{min} = 0.2$; the dashed curve relates to the ideal optimum fin for $u_{min} = 0$.

and using condition (4), we find that a fin can be considered narrow when the following conditions are satisfied:

$$\left[\frac{Q_0}{2\lambda(T_0 - T_m)}\frac{du}{dz}\right]^2 \ll 1; \quad \frac{du}{dz} = 2, \ \theta_l = 0;$$

$$\frac{Q_0}{\lambda^2(T - T_m)^2} \ll 1.$$

The higher the thermal conductivity of the fin material, the stricter is the assumption of smallness of y'. If $\theta_l > 0$, then $du/dz \to \infty$ when $q \to 0$, which means that at the tip the fin contour becomes vertical. However, if $\theta_l = 0$, then for $q \to 0$ $du/dz = 0$ also. The drawback of such a fin is the simultaneous vanishing of du/dz and of the half-thickness u, although from the point of view of its cross-sectional area this fin is the absolute optimum. The very sharp tip of the fin makes it technologically impossible.

The necessity to make the end part of the fin of thickness $2u_{min}$ leads to a fin of composite contour. The curvilinear part of the fin up to $u = u_{min}$ is calculated from formulas (19) and (20), and its rectilinear part from known heat transfer relationships with the following boundary conditions (see Fig. 1a):

$$z_1 = 0, \quad q = q^*, \quad \theta = \theta^*; \quad z_1 = l_1, \quad q = 0.$$

The length of the fin rectilinear part is determined as follows:

$$l_1 = -\frac{\sqrt{u_{min}}}{2}\ln\frac{\theta^* - q^*/\sqrt{u_{min}}}{\theta^* + q^*/\sqrt{u_{min}}}. \tag{23}$$

The heat flux q* is determined from relationship (20) by setting $u = u_{min}$, and θ_l is the temperature which would obtain at the tip of a sharp fin without an additional rectilinear part.

Temperature θ^* is obtained by setting the heat flux $q = q^*$ in formula (18). The temperature of the fin rectilinear tip can be obtained from formula

$$\theta_l^* = \theta^*\sqrt{\frac{\theta^* - q^*/\sqrt{u_{min}}}{\theta^* + q^*/\sqrt{u_{min}}}}. \tag{24}$$

The over-all length of the fin is equal to the sum of the lengths of the curvilinear and rectilinear parts. These magnitudes calculated by the formulas given here are shown in Fig. 3, from which it is seen that the length of a fin decreases with increasing θ_l at a rate dependent on u_{min}. The length of a fin with a narrow tip increases faster than that of a thick fin. This increase is due to the lengthening of the curvilinear part of the fin. A judicious selection of temperature θ_l and due consideration to the increase of thickness u at the fin root and of its length l are required in order to reduce the over-all dimensions of the fin. Temperature θ_l also affects the fin cross-sectional area f, as shown by

$$f = \frac{8}{3}\frac{1 - q^{*3/2}}{1 - \theta_l^3} + 2u_{min}l_1. \tag{25}$$

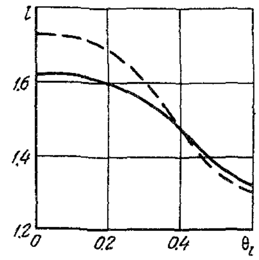

Fig. 3. Dependence of fin length on temperature θ_l. The solid line relates to $u_{min} = 0.5$, and the dashed line to $u_{min} = 0.2$.

The dependence of f on temperature θ_l is shown in Fig. 2. It is seen that the cross-sectional area has a mildly sloping minimum in a wide range of temperatures θ_l, and that this minimum is dependent on the thickness u of the fin tip. The smaller the thickness u_{min}, the smaller the area f and the lower the temperature θ_l at which the mildly sloping minimum is attained. The curves in Figs. 2 and 3 show that a reduction of thickness u_{min} leads to a decrease of the area, but for sufficiently small θ_l results in a longer fin (see Fig. 3 for the effect of changing from $u_{min} = 0.5$ to $u_{min} = 0.2$). When selecting θ_l for a specific case, preference is to be given to obtaining a fin of either minimum cross-sectional area, or minimum length l.

Temperature distribution along a sharp fin for various temperatures θ_l is given in Fig. 4, which shows that for θ_l decreasing to zero this distribution tends to be linear. The cross-sectional area of such fins is the smallest, but the fin is very sharp. Actual fins of such sharpness cannot be produced. Hence linear distribution of temperature does not obtain in these.

NOTATION

Q is the heat flux; λ is the coefficient of thermal conductivity; y is the fin half-width; T is the temperature of the fin surface; T_0 is the temperature of the fin root; T_m is the temperature of the medium; α is the heat-exchange coefficient; x is the coordinate along the fin center line; F is the area of the fin cross section; q is the dimensionless heat flux; f is the dimensionless area; z is the dimensionless coordinate along the fin center line; u is the dimensionless half-width of the fin; θ is the dimensionless temperature; H is the Hamilton function; ψ_1 and ψ_2 are auxiliary functions; l_1 is the dimensionless length of the fin rectilinear part; q* is the heat flux at the junction of the two parts

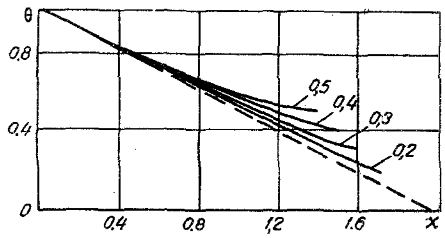

Fig. 4. Temperature distribution along the length of the fin. Numerals indicate values of θ_l.

of the fin profile; θ_l^* is the temperature of the tip of the rectilinear part of a fin; and θ_l is the temperature of the sharp tip of a curvilinear fin.

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